USA Mathematical Talent Search

Round 2 Solutions

Year 37 — Academic Year 2025–2026

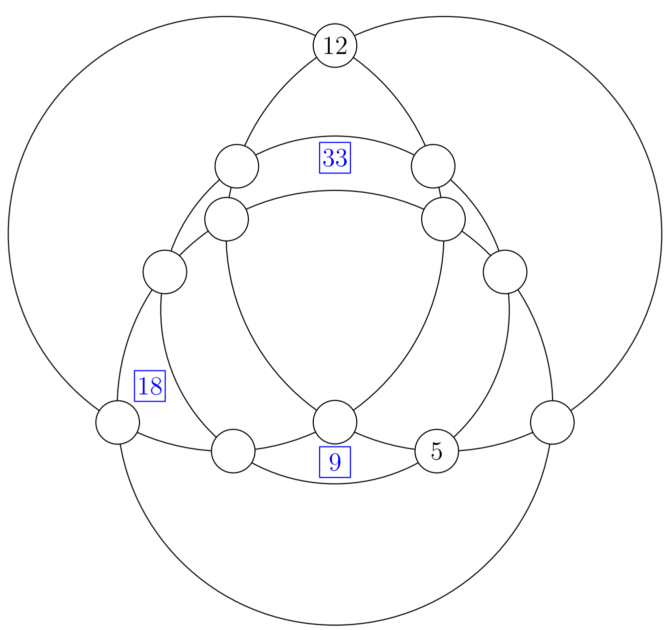
www.usamts.org

1/2/37. Place the whole numbers $1 - 12$ in the smaller circles appearing at the intersection points of the four larger circles below so that:

- Each whole number $1 - 12$ appears exactly once.
- The sum of the 6 numbers around any of the four larger circles is some constant C.
- If two intersection points are adjacent (meaning they are connected by an arc which does not contain other intersection points), then the numbers placed at those vertices are non-consecutive.
- In some regions of the figure there are squares that contain numbers. The sum of the numbers placed in the smaller circles around such regions must be equal to the value inside the square.

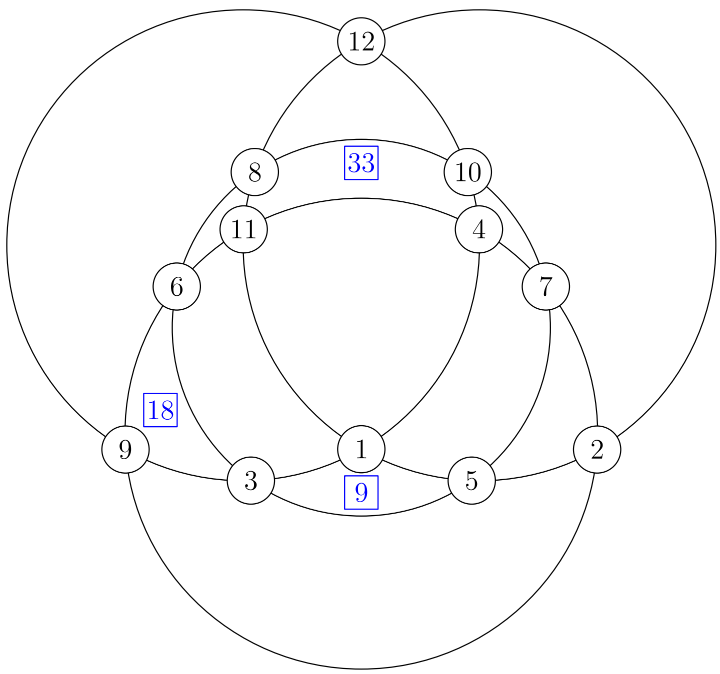
The positions of the numbers 5 and 12 have been given for you.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



Solution

The solution can be obtained by converting some of the conditions into sets of equations, and then using other conditions, plus some mild case work, to rule out possible entries.



2/2/37. We call a positive integer $n > 4$ *amazing* if, for any list ℓ consisting of $n-4$ copies of the number 1 and two copies of the number 2, it is always possible to divide ℓ into *at least two* proper contiguous sub-lists which have equal sum.

For example, $n = 6$ is amazing, since we have the following lists with divisions indicated by vertical bars.

- $(1, 1 \,|\, 2 \,|\, 2)$
- $(1, 2 \,|\, 1, 2)$
- $(1, 2 \,|\, 2, 1)$
- $(2, 1 \,|\, 1, 2)$
- $(2, 1 \,|\, 2, 1)$
- $(2 \,|\, 2 \,|\, 1, 1)$

Find, with proof, all amazing positive integers.

Solution

The answer is that $n = 6$ or n has at least three distinct prime factors. First, suppose $n > 6$ and n has at most two distinct prime factors.

If $n = p^e$ has only one prime factor, then consider the list

$$(\underbrace{1, 1, \ldots, 1}_{\frac{n}{p}-1}, 2, 2, 1, \ldots, 1)$$

consisting of $\frac{n}{p} - 1$ copies of the number 1 before both instances of the number 2. Suppose we can divide this list into contiguous sublists with common sum d. Then $d \mid \frac{n}{p}$, so some number of our sublists can be concatenated from the left to give a sum of $\frac{n}{p}$. But it is impossible to write down an initial sublist of this list with sum $\frac{n}{p}$. So this list cannot be divided, which means that n is not amazing.

Next, suppose n is divisible by both p and q, where p and q are distinct primes, but $n \neq 6$. Suppose without loss of generality that $p < q$. Then since $n \neq 6$, we have $q \geq 5$ and $n \geq 10$. So $\frac{3}{10} \cdot n - 2 > 0$, which implies that $n - 2 > \frac{7}{10} \cdot n$. Therefore

$$\left(\frac{n}{p} - 1\right) + \left(\frac{n}{q} - 1\right) + 2 \leq \frac{n}{2} + \frac{n}{5} \leq \frac{7}{10} \cdot n < n - 2.$$

So it makes sense to consider the list

$$(\underbrace{1,\ldots,1}_{\frac{n}{p}-1},2,1,\ldots,1,2,\underbrace{1,\ldots,1}_{\frac{n}{q}-1}).$$

Now suppose that this list can be divided, with common sum d. Then d either divides $\frac{n}{p}$ or $\frac{n}{q}$. But in the first case, we should be able to concatenate some number of our sublists from the left to obtain a total sum of $\frac{n}{p}$, which is impossible. Similarly, in the second case, we should be able to concatenate some number of our sublists from the right to obtain a sum of $\frac{n}{q}$, which is impossible. We conclude that this list is not dividable, so n is not amazing.

Conversely, suppose n is divisible by three distinct primes, say p, q, and r. Let ℓ be an arbitrary list with $n-4$ 1's and two 2's. Consider a new list ℓ' which looks as follows:

$$\ell' = (1,\ldots,1,\hat{1},\hat{1},1,\ldots,1,\hat{1},\hat{1},1\ldots,1)$$

where the symbols $\hat{1}$ are inserted exactly where the 2's were in ℓ. Suppose the first $\hat{1}$ occurs at position a, where we start with index 1 at the left. Next, suppose that the **third** $\hat{1}$ occurs at position b from the left. Note that $b \le n-1$.

Now, note that it is impossible for a multiple of two of $\frac{n}{p}$, $\frac{n}{q}$, $\frac{n}{r}$ to be less than n (for if x is divisible by both, say, $\frac{n}{p}$ and $\frac{n}{q}$, then x must be divisible by n itself).

So we know that a is divisible by **at most one of** $\frac{n}{p}$, $\frac{n}{q}$, or $\frac{n}{r}$. The same statement holds for b. So suppose without loss of generality that neither a nor b is divisible by $\frac{n}{r}$.

But this means that ℓ' can be subdivided into contiguous parts of size $\frac{n}{r}$ where no dividing line ends up right after position a or position b. Such a division can be naturally extended to a division of ℓ itself, by placing the dividing lines in the exact same places.

We conclude that ℓ is divisible, and since ℓ was arbitrary, n must be amazing.

3/2/37. Let $\mathbb{Z}^+$ denote the set of positive integers. Determine, with proof, whether there exist functions $f, g : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(g(g(x))) = 3x,$$

$$g(f(g(x))) = 5x,$$

and

$$g(g(f(x))) = 7x$$

for all $x \in \mathbb{Z}^+$.

Solution

Yes, such functions f and g do exist. Here is one example.

Given a positive integer x, let a, b, and c be the exponents of 3, 5, and 7 in the prime factorization of x, respectively, and write

$$x = 3^a 5^b 7^c y$$

(so y is an integer that is not divisible by 3, 5, or 7). We then define

$$f(x) = 3^{c+1} 5^a 7^b y$$

and

$$g(x) = 3^c 5^a 7^b y.$$

In other words, g cyclically permutes the exponents on 3, 5, and 7 in the prime factorization of its input, while f cyclically permutes the exponents and then also increases the exponent on 3 by 1. To check that these functions work, we have

$$\begin{aligned} f(g(g(x))) &= f(g(g(3^a 5^b 7^c y))) \\ &= f(g(3^c 5^a 7^b y)) \\ &= f(3^b 5^c 7^a y) \\ &= 3^{a+1} 5^b 7^c y \\ &= 3x, \end{aligned}$$

$$\begin{aligned} g(f(g(x))) &= g(f(g(3^a 5^b 7^c y))) \\ &= g(f(3^c 5^a 7^b y)) \\ &= g(3^{b+1} 5^c 7^a y) \\ &= 3^a 5^{b+1} 7^c y \\ &= 5x, \end{aligned}$$

and

$$\begin{aligned} g(g(f(x))) &= g(g(f(3^a5^b7^cy))) \\ &= g(g(3^{c+1}5^a7^by)) \\ &= g(3^b5^{c+1}7^ay) \\ &= 3^a5^b7^{c+1}y \\ &= 7x. \end{aligned}$$

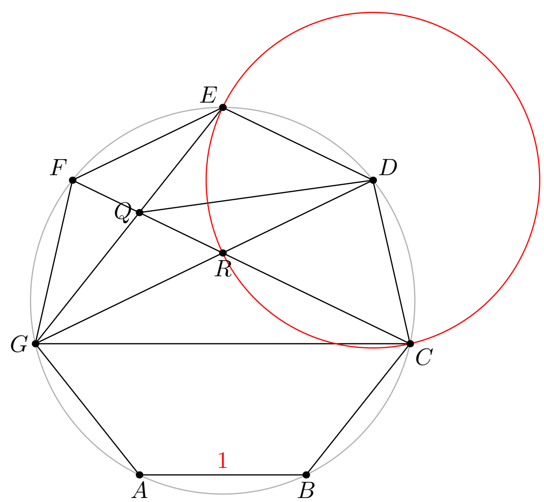
4/2/37. Let $ABCDEFG$ be a regular heptagon with side length 1. Let P be the point on $\overline{EF}$ such that $\angle PAB = 90^\circ$. Compute lengths PC and PD.

Synthetic Solution: We claim that $PC = 2$ and $PD = \sqrt{2}$. We establish these results with a series of claims.

Let $\theta = \frac{180^\circ}{7}$. (Note that the angles of a triangle add up to $7\theta = 180^\circ$.)

Claim 1. Let $\overline{CF}$ and $\overline{EG}$ intersect at Q. Then $DQ = \sqrt{2}$.

Proof. Let $\overline{CF}$ and $\overline{DG}$ intersect at R.



Since $\overline{FR}$ is parallel to $\overline{ED}$, and $\overline{DR}$ is parallel to $\overline{EF}$, we have that $FEDR$ is a parallelogram, so $DR = EF = 1$. Thus, E, R, and C all lie on the circle centered at D with radius 1.

We know that all the vertices of $ABCDEFG$ lie on a circle. (Henceforth, all references to arcs will be with respect to this circle.) Then $\angle QGR = \frac{\text{arc } ED}{2} = \theta$ and $\angle QCG = \frac{\text{arc } FG}{2} = \theta$, so triangles QGR and QCG are similar. Then

$$\frac{QR}{QG} = \frac{QG}{QC},$$

so $QR \cdot QC = QG^2$.

Note that $\angle QFG = \frac{\text{arc } CG}{2} = 3\theta$, and

$$\angle FQG = \frac{\text{arc } FG + \text{arc } EC}{2} = 3\theta,$$

so triangle FQG is isosceles with $GQ = GF = 1$. Hence, $QR \cdot QC = 1$.

Finally, by power of a point, $QD^2 - 1 = QR \cdot QC = 1$, so $QD = \sqrt{2}$.

Claim 2. $DP = \sqrt{2}$.

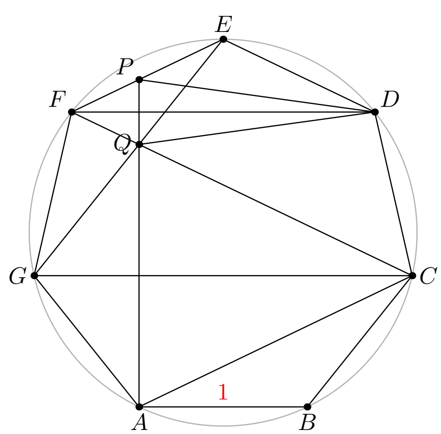
Proof. We see that

$$\angle QGC = \frac{\text{arc } EC}{2} = \frac{\text{arc } AC}{2} = \angle AGC$$

and

$$\angle QCG = \frac{\text{arc } FG}{2} = \frac{\text{arc } AG}{2} = \angle ACG,$$

so triangles QGC and AGC are congruent. Hence, A and Q are reflections of each other across $\overline{GC}$, which means $\overline{QA}$ is perpendicular to $\overline{GC}$. But $\overline{GC}$ is parallel to $\overline{AB}$, so $\overline{QA}$ is also perpendicular to $\overline{AB}$. Therefore, Q lies on $\overline{PA}$.



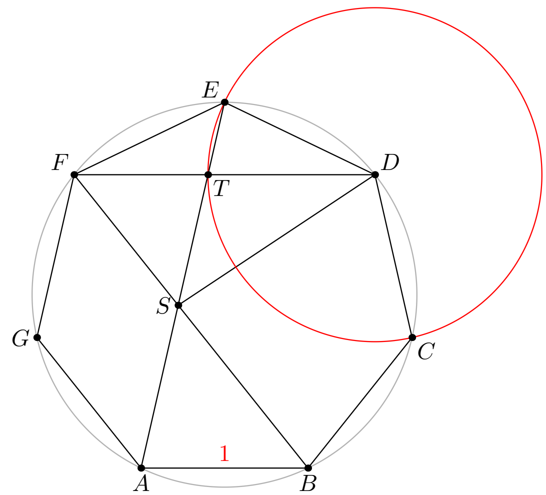
Furthermore,

$$\angle PFD = \frac{\text{arc } ED}{2} = \frac{\text{arc } CD}{2} = \angle QFD,$$

and $\overline{PQ}$ is perpendicular to $\overline{FD}$, so P and Q are reflections of each other across $\overline{FD}$. Therefore, $DP = DQ = \sqrt{2}$.

Claim 3. Let $\overline{AE}$ and $\overline{BF}$ intersect at S. Then $DS = \sqrt{2}$.

Proof. Let $\overline{AE}$ and $\overline{DF}$ intersect at T.



Since $\overline{DF}$ is parallel to $\overline{AB}$, $\angle DTE = \angle BAE$. Since arcs BE and AD are equal, $\angle BAE = \angle DEA$. Hence, $\angle DTE = \angle DET$, so triangle DET is isosceles with $DE = DT$. Thus, E, T, and C all lie on the circle centered at D with radius 1.

Note that

$$\angle FSE = \frac{\text{arc } EF + \text{arc } AB}{2} = 2\theta,$$
$$\angle FES = \frac{\text{arc } AF}{2} = 2\theta,$$
$$\angle SFT = \frac{\text{arc } BD}{2} = 2\theta.$$

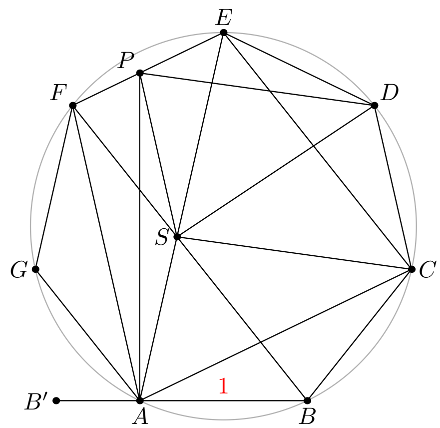
This means triangles FSE and TSF are isosceles and similar, so

$$\frac{FS}{TS} = \frac{SE}{FS}.$$

Then $ST \cdot SE = SF^2 = EF^2 = 1$. By power of a point, $SD^2 - 1 = ST \cdot SE = 1$, so $SD = \sqrt{2}$. ■

Claim 4. $CP = 2$.

Proof. We have shown that $DS = \sqrt{2}$. By symmetry (say, reflecting over the perpendicular bisector of $\overline{CD}$), we also have that $CS = \sqrt{2}$.



Extend $\overline{AB}$ past A to a point B'. We have that $\angle BAE = \frac{\text{arc } BE}{2} = 3\theta$ and $\angle EAF = \frac{\text{arc } EF}{2} = \theta$, so $\angle B'AF = 180^\circ - \angle BAE - \angle EAF = 180^\circ - 3\theta - \theta = 3\theta = \angle BAE$. Hence, $\overline{AP}$ bisects $\angle EAF$, so by the Angle Bisector Theorem,

$$\frac{FP}{EP} = \frac{AF}{AE} = \frac{AC}{AE}.$$

Since $\overline{EF}$ is parallel to $\overline{AC}$, and $\overline{BF}$ is parallel to $\overline{CE}$, the corresponding sides of triangles EFS and ACE are similar, so

$$\frac{SF}{SE} = \frac{CE}{AE} = \frac{AC}{AE} = \frac{FP}{EP}.$$

Then by the Angle Bisector Theorem, $\overline{SP}$ bisects $\angle FSE$.

We can compute that $\angle FSE = 2\theta$, so $\angle FSP = \theta$. We can also compute that $\angle PFS = 3\theta$, so $\angle FPS = 180^\circ - \angle FSP - \angle PFS = 180^\circ - \theta - 3\theta = 3\theta$, so triangle FSP is isosceles with $SP = SF = 1$.

We have shown that $DP = CS = \sqrt{2}$ and $CD = SP = 1$, so $CSPD$ is a parallelogram. Then by the Parallelogram Law,

$$2(CS^2 + SP^2) = CP^2 + SD^2.$$

This gives us $2(2+1) = CP^2 + 2$, so $CP = 2$.

Complex Number Solution:

We place the diagram in the complex plane. Let $\omega = e^{2\pi i/7}$. Then $\omega^7 = 1$, so $\omega^7 - 1 = 0$. This factors as

$$(\omega - 1)(\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0.$$

Since $\omega \neq 1$, we have that

$$\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0.$$

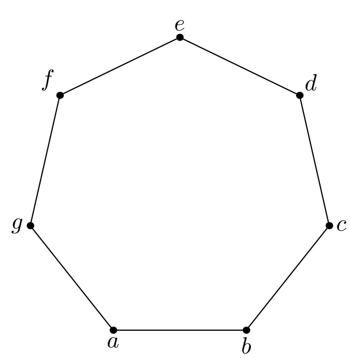
Also, since $|\omega| = 1$, the conjugate of any power of ω is its reciprocal. For example,

$$\overline{\omega^2} = \frac{1}{\omega^2} = \omega^5.$$

Let

$$\begin{aligned} a &= 0, \\ b &= 1, \\ c &= 1 + \omega, \\ d &= 1 + \omega + w^2, \\ e &= 1 + \omega + w^2 + w^3 = -\omega^6 - \omega^5 - \omega^4, \\ f &= -\omega^6 - \omega^5, \\ g &= -\omega^6. \end{aligned}$$

Then a, b, c, d, e, f, and g form the vertices of a regular heptagon with side length 1.



The point P lies on $\overline{EF}$ so that $\angle PAB = 90°$. The corresponding complex number p lies on the line containing e and f, so

$$p = -\omega^6 - \omega^5 + t\omega^4$$

for some real number t. Also, p lies on the imaginary axis, so $\overline{p} = -p$.

Then

$$-\omega - \omega^2 + t\omega^3 = \omega^6 + \omega^5 - t\omega^4.$$

Solving for t, we find

$$\begin{aligned} t &= \frac{\omega + \omega^2 + \omega^5 + \omega^6}{\omega^3 + \omega^4} \\ &= \frac{\omega(1+\omega)(1+\omega^4)}{\omega^3(1+\omega)} \\ &= \frac{1+\omega^4}{\omega^2} = \omega^5 + \omega^2, \end{aligned}$$

and

$$p = -\omega^6 - \omega^5 + (\omega^2 + \omega^2)\omega^4 = \omega^2 - \omega^5.$$

Then

$$p - c = (\omega^2 - \omega^5) - (1 + \omega) = -1 - \omega + \omega^2 - \omega^5,$$

so

$$\begin{aligned} |p-c|^2 &= (p-c)(\overline{p-c}) \\ &= (-1 - \omega + \omega^2 - \omega^5)(-1 - \omega^6 + \omega^5 - \omega^2) \\ &= 1 + \omega^6 - \omega^5 + \omega^2 \\ &\quad + \omega + \omega^7 - \omega^6 + \omega^3 \\ &\quad - \omega^2 - \omega^8 + \omega^7 - \omega^4 \\ &\quad + \omega^5 + \omega^{11} - \omega^{10} + \omega^7 \\ &= 1 + \omega^6 - \omega^5 + \omega^2 \\ &\quad + \omega + 1 - \omega^6 + \omega^3 \\ &\quad - \omega^2 - \omega + 1 - \omega^4 \\ &\quad + \omega^5 + \omega^4 - \omega^3 + 1 \\ &= 4. \end{aligned}$$

Hence, $PC = 2$.

Also,

$$p - d = (\omega^2 - \omega^5) - (1 + \omega + \omega^2) = -1 - \omega - \omega^5,$$

so

$$\begin{aligned}
|p-d|^2 &= (p-d)(\overline{p-d}) \\
&= (-1-\omega-\omega^5)(-1-\omega^6-\omega^2) \\
&= 1+\omega^6+\omega^2 \\
&\quad +\omega+\omega^7+\omega^3 \\
&\quad +\omega^5+\omega^{11}+\omega^7 \\
&= 1+\omega^6+\omega^2 \\
&\quad +\omega+1+\omega^3 \\
&\quad +\omega^5+\omega^4+1 \\
&= 2.
\end{aligned}$$

Hence, $PD = \sqrt{2}$.

5/2/37. In a round-robin tournament with 1000 teams, each team plays one game against each other team, and each game either results in one of the two teams winning, or a draw. Show that at least one of the following statements must be true:

- There exists a team that draws against at least 10 other teams.
- There exists a group of exactly 10 teams that can be numbered from 1 to 10 such that for all $1 \leq i < 10$, team i won against team $i+1$, and team 10 won against team 1.
- There exists a group of exactly 10 teams that can be numbered from 1 to 10 such that if two teams are numbered i and j with $i < j$, then team i won against team j.

Solution 1.

We first show that either the first condition holds, or we can find a large set of teams with no draws amongst them.

Claim. *Either there exists a team that draws against at least* 10 *other teams, or there exists a set of* 100 *teams, no two of which draw against each other.*

Proof. Suppose that no team draws against at least 10 other teams. We recursively construct a sequence of teams $t_1, \ldots, t_{100}$, no two of which draw against each other.

To start, let t_1 be any team. Having chosen $t_1, \ldots, t_k$ for $k < 100$, note that there are at most $9k$ teams that draw against any of $t_1, \ldots, t_k$, since each team draws against at most 9 teams. Since $k + 9k = 10k < 1000$, this means there exists a team that is not equal to $t_1, \ldots, t_k$ and also does not draw against any of them. We pick such a team and let it be t_{k+1}.

By construction, then, the teams $t_1, \ldots, t_{100}$ are all distinct, and no two draw against each other. Thus we have a set of 100 teams, no two of which draw against each other. □

From now on, we assume that we have a set of 100 teams, no two of which draw against each other, and restrict our attention to only these 100 teams. We now analyze the structure of round-robin tournaments without draws more generally. Given a round-robin tournament, if $t_1, \ldots, t_k$ are distinct teams such that t_i beats t_{i+1} for $i = 1, \ldots, k-1$ and t_k beats t_1, we will say that $(t_1, \ldots, t_k)$ is a *k-cycle* in the tournament.

Lemma 1. *Suppose that a round-robin tournament with n teams and no draws has no k-cycles for any k. Then the teams can be numbered from* 1 *to n such that team i beats team j for all $i < j$.*

Proof. We use induction on n. The base case $n = 0$ is trivial.

Now suppose $n > 0$, and the result is known for all tournaments with $n - 1$ teams. We claim there exists a team t that loses all of their games. Indeed, suppose that no such team

t exists. Then we can start with any team t_1, and recursively choose teams t_{i+1} such that t_i beats t_{i+1}, since every team beats some other team. Since there are only finitely many teams, this sequence of teams must eventually repeat. If j is minimal such that t_j is equal to t_i for some $i < j$, then $(t_i, t_{i+1}, \ldots, t_{j-1})$ is a cycle, contradicting the assumption that our tournament has no cycles.

Thus there exists a team t that loses all their games. Removing t from the tournament, we get a tournament with $n-1$ teams, and by the induction hypothesis we can label its teams from 1 to $n-1$ such that team i beats team j for all $i > j$. Labelling t as team n, we then get such a labelling of our n-team tournament. □

Lemma 2. *In a round-robin tournament without draws, suppose $(t_1, \ldots, t_k)$ is a cycle of maximal length. Then for any team $u \neq t_1, \ldots, t_k$, either u beats t_i for all i, or u loses to t_i for all i.*

Proof. Suppose that u beats t_i for some i. Then u must also beat t_{i-1} (or t_k if $i = 1$), since otherwise we could insert u in between t_{i-1} and t_i in our cycle to get a longer cycle. Repeating this argument with t_{i-1} in place of t_i, we learn that u must also beat t_{i-2}, and similarly by induction we see that u beats every team in our cycle. Thus if u beats any team in the cycle, it must beat every team in the cycle, so either u beats every t_i or u loses to every t_i. □

Lemma 3. *For any round-robin tournament without draws, the teams can be partitioned into sets $S_1, \ldots, S_m$ which satisfy the following properties:*

1. *For all $i < j$, if $t \in S_i$ and $u \in S_j$, then t beats u.*
2. *For each i, either $|S_i| = 1$ or all the teams in S_i can be arranged in a cycle.*

Proof. We use strong induction on the number of teams in the tournament. Suppose that our tournament has n teams, and we know the result is true for all tournaments with less than n teams.

If our tournament has no cycles, then we are done by Lemma 1, since we can choose each set S_i to have just a single element.

If our tournament has a cycle, choose a cycle $(t_1, \ldots, t_k)$ of maximal length in the tournament, and let $S = \{t_1, \ldots, t_k\}$. By Lemma 2, each team not in S either beats every team in S or loses to every team in S. Let A be the set of teams that beat every team in S and let B be the set of teams that lose to every team in S. If $a \in A$ and $b \in B$, then a must beat b, since otherwise $(a, t_1, \ldots, t_k, b)$ would be a cycle, contradicting the maximality of $(t_1, \ldots, t_k)$.

Applying the induction hypothesis to A and B, we can partition A into sets $S_1, \ldots, S_r$ and partition B into sets $S_{r+2}, \ldots, S_m$ satisfying properties (a) and (b) in the statement of the Lemma. Defining $S_{r+1} = S$, the sequence $S_1, \ldots, S_m$ then also has these properties for our original tournament, since each team in A beats each team in S, each team in S beats each team in B, and each team in A beats each team in B. □

Applying Lemma 3 to our set of 100 teams, no two of which draw, we can partition these 100 teams into sets $S_1, \ldots, S_m$ satisfying properties (1) and (2). If $m \geq 10$, then by picking one team from each of $S_1, \ldots, S_{10}$, we obtain a set of 10 teams satisfying the third condition of the problem, so we may assume that $m < 10$. It follows that for some i, we must have $|S_i| > 10$. That is, our tournament has a k-cycle for some $k > 10$. To conclude that we have a 10-cycle and thus the second condition of the problem is satisfied, we use the following lemma.

Lemma 4. *Suppose a round-robin tournament without draws has a k-cycle. Then the tournament also has an ℓ-cycle for each ℓ such that $3 \leq \ell \leq k$.*

Proof. Using descending induction on ℓ, it suffices to show that if $k > 3$ and our tournament has a k-cycle $(t_1, \ldots, t_k)$, then it has a $(k-1)$-cycle. Let $S = \{t_1, \ldots, t_k\}$.

First, suppose that no proper subset of S can be arranged into a cycle. By Lemma 1 applied to $\{t_1, t_2, t_3\}$, it follows that t_1 beats t_3. But now by Lemma 1 applied to $S \setminus \{t_2\}$, it follows that t_1 beats t_k, since the only possible ordering of $S \setminus \{t_2\}$ satisfying the conclusion of Lemma 1 is $t_1, t_3, t_4, \ldots, t_k$. This contradicts our assumption that $(t_1, \ldots, t_k)$ was a cycle.

Thus there is a proper subset of S that can be arranged into a cycle. Let T be a proper subset of S of maximal size that can be arranged into a cycle. If $|T| = k - 1$, we are done, so suppose $|T| < k - 1$. For each $u \in S \setminus T$, note that $T \cup \{u\}$ cannot be arranged into a cycle, so by Lemma 2, either u beats every element of T or u loses to each element of T. Say that $u \in S \setminus T$ is *strong* if u beats every element of T and *weak* if u loses to every element of T.

By cyclically permuting $t_1, \ldots, t_k$, we may assume that $t_1 \in T$ and $t_2 \notin T$. Then t_2 must be weak, since it loses to t_1. Let $i > 1$ be minimal such that $t_i \in T$. Then t_{i-1} must be strong, since it beats t_i. Since t_2 is weak and t_{i-1} is strong, and $t_j \in S \setminus T$ for all $j \in \{2, \ldots, i-1\}$, there must exist some $j \in \{2, \ldots, i-2\}$ such that t_j is weak and t_{j+1} is strong. Then $(T \setminus \{t_1\}) \cup \{t_j, t_{j+1}\}$ can be arranged into a cycle, by replacing t_1 with (t_j, t_{j+1}) (in that order) in the cycle arrangement of T. This contradicts the maximality of T. □

To summarize, in our original round-robin tournament of 1000 teams, we must either have a team that draws against at least 10 other teams (so that the first condition in the problem is true), or a group of 100 teams such that no two of these teams draw. In the latter case, we apply Lemma 3 to those 100 teams to either obtain a set of 10 teams satisfying the third condition in the problem (if $m \geq 10$) or a cycle of length greater than 10 (if $m < 10$), which by Lemma 4 then also gives a 10-cycle, satisfying the second condition in the problem. This means that at least one of our three conditions must be true, as desired.

Solution 2 (using the language of Graph Theory). We convert this to a graph theory problem. First, build a red-blue edge-colored K_{1000} where the 1000 teams are vertices, and the edge between two teams is colored red if the game drew and blue if the game had a winner.

The first of our conditions is equivalent to the existence of a red star with 10 leaves (i.e. a red $K_{1,10}$) in our graph. To remove this condition from our problem, we show that if a red-blue edge-colored K_n has $r(b-1)+1$ vertices, it must either contain a red star with r leaves (i.e. a red $K_{1,r}$) or a blue K_b.

- Pick a vertex v_1. If it has at least r red incident edges we're done. Else there are at least $r(b-2)+1$ blue incident edges; let the set of corresponding neighbors be S_1.
- Now pick $v_2 \in S_1$. If it has at least r red incident edges to other vertices in S_1 we're done. Else there are at least $r(b-3)+1$ blue incident edges within S_1; let the set of corresponding neighbors be S_2.
- Repeat this argument until we run out of vertices. If we aren't done by the time we run out of vertices then $v_1, v_2, \ldots, v_b$ will form a complete K_b, since $K_i \subset K_j$ for all $i > j$.

Applying this to our K_{1000}, it follows that we must either have a red star with 10 leaves (so that our first condition is true), or a blue K_{100}, which means there's a group of 100 teams such that each match between two teams in the group resulted in one team winning.

Now, we show that in the latter case, one of the last two conditions must be true: Given 100 teams in our tournament such that each match amongst them resulted in one team winning, build a digraph so that vertex represents a team, and there is exactly one directed edge between any two teams pointing from the winning team to the losing team in the corresponding match. (Such a digraph is known as a *tournament*; henceforth, our use of this term will be graph-theoretic unless otherwise noted.) Then the second condition corresponds to a directed cycle of length exactly 10 in our digraph, and the third condition corresponds to a transitive subtournament with exactly 10 vertices in our digraph.

We claim that in a tournament with k^2+1 vertices, there must exist a directed cycle with exactly $k+1$ vertices or a transitive subtournament with exactly $k+1$ vertices. Setting $k=9$ then finishes the problem. The key to showing this is the following lemma:

Lemma: If a tournament has no directed cycle of length n, it also does not have a directed cycle of length $n+1$.

Proof: Suppose for a contradiction that a directed $(n+1)$-cycle C exists. Take the maximal non-Hamiltonian cycle in $V(C)$; suppose this is C', and has vertices $v_1, \ldots, v_k$ in that order. By hypothesis, this has length at most $n-1$, so there are at least two vertices in $V(C) \setminus V(C')$.

Let $v \in V(C) \setminus V(C')$. If there are two consecutive vertices $v_i \to v_{i+1}$ in $V(C)$ such that $v_i \to v$ and $v \to v_{i+1}$, we can extend C' to get a larger non-Hamiltonian cycle, a contradiction. Hence we must either have $v \to w$ for all $w \in V(C')$ or $w \to v$ for all $w \in V(C')$. In the first case, call v strong, and in the second case, call v weak. Then every strong vertex must point to every weak vertex: Suppose not. Then if v is strong and w is weak with $w \to v$, the cycle $v \to v_1 \to \cdots \to v_{k-1} \to w \to v$ is not Hamiltonian but longer than C', a contradiction.

But since there are at least two vertices in $V(C) \setminus V(C')$, it now follows that the induced subtournament on $V(C)$ can't be Hamiltonian:

- If there are two strong vertices, we can't get back to the set of strong vertices after leaving it, preventing us from creating a Hamiltonian cycle.
- If there are two weak vertices, we can't leave the set of weak vertices after entering it, preventing us from creating a Hamiltonian cycle.
- If there's exactly one strong and one weak vertex, we can't reach the strong vertex from the weak vertex, preventing us from creating a Hamiltonian cycle.

That's a contradiction, so our tournament cannot contain a directed cycle of length $n+1$. ■

Now, we use our lemma to prove our claim. Suppose a tournament T with k^2+1 vertices does not contain a directed cycle with exactly $k+1$ vertices. By our lemma, T can't contain any longer directed cycle. Take a maximal directed cycle C in T (where we allow one-vertex cycles); by hypothesis C contains at most k vertices. By the structural argument in the proof of our lemma above, each vertex $v \in V(T) \setminus V(C)$ is either strong (i.e. $v \to w$ for all $w \in V(C)$) or weak (i.e. $w \to v$ for all $w \in V(C)$). By iteratively repeating this process on the subtournaments induced by our strong and weak vertices, we eventually partition our tournament into sets of vertices $S_1, S_2, \ldots, S_m$ such that

- There is a Hamiltonian cycle within each $V(S_i)$;
- For all $1 \le i \le j \le m$, if $v \in S_i$ and $w \in S_j$, then $v \to w$. (This follows from transitively applying the fact that strong vertices point to weak vertices in the lemma argument.)

Since each S_i contains at most k vertices, it follows that $m \ge k+1$. Hence choosing one vertex from each of $S_1, S_2, \ldots, S_{k+1}$ induces a transitive subtournament on $k+1$ vertices.

To summarize, in our original round-robin tournament of 1000 teams, we must either have a team that draws against at least 10 other teams (so that the first condition is true), or a group of 100 teams such that each match between two teams in the group resulted in one team winning. In the latter case, we know that if we take any 82 of our 100 teams, the corresponding graph-theoretic tournament either has a directed cycle of length 10 (so that our second condition is true), or a transitive subtournament with 10 teams (so that the third condition is true). This means that at least one of our three conditions must be true, as desired.