USA Mathematical Talent Search

Round 2 Grading Rubric

Year 33 — Academic Year 2021–2022

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/2/33:

Award **5 points** for the correct configuration of numbers. No justification is required. In general, withhold **1 point** for each incorrect entry, but note the specific situations discussed below.

Award a total score of **1 point** if the student has at least one incorrect shaded entry, but the configuration is a Latin Square.

Award **2 points** if the student fills in all the shaded entries correctly, but doesn't fill in the rest of the grid. Award **0 points** if any of the shaded entries are incorrect and the student doesn't fill in the rest of the grid.

Problem 2/2/33:

Note: The two main approaches to this problem were to (1) find the two counts, or (2) establish a bijection between the two counts.

Note: Because there are only four possible answers, the answer itself (without supporting explanation) isn't sufficient to get any points.

Student finds the two counts:

Award **5 points** if the student finds both counts with proof and award **2 points** if the student finds one of the counts with proof. Given that finding one of the counts doesn't tell us anything about the relative size of the two counts, this is less than halfway to solving the problem. Award partial credit as appropriate for significant constructive progress towards finding the counts.

Note: If there were major errors in a student's counting method, we typically did not award partial credit unless the student had significant constructive progress towards a correct count. A common error was to pre-assign the 0 and 7 when counting the n-tuples, which led to the incorrect counting of n-tuples containing more than one 0 or 7, such as $(0, 0, 7)$.

Student establishes a bijection between the two counts:

3 points: Student finds a good way to relate the counts in the two conditions (e.g., sets up a Venn diagram and assigns numbers to regions as appropriate), and makes some effort to connect the setup to the two conditions. Award partial credit as appropriate for significant constructive progress towards the setup and/or if aspects of the student's explanation are unclear.

1 point: Student shows that any n-tuple satisfying the first condition corresponds to an ordered triple satisfying the second condition.

1 point: Student shows that any ordered triple satisfying the second condition corresponds to an n-tuple satisfying the first condition.

Problem 3/2/33:

Note: Students used a variety of algebraic manipulations. For students who did not arrive at the correct answer, award partial credit as appropriate for constructive algebraic manipulations. A point allocation for the official solution is as follows.

1 point: Student notes that

$$\left(\sqrt{a^2+1}-a\right)\left(a+\sqrt{a^2+1}\right) = (a^2+1)-a^2 = 1$$

for all real a.

1 point: Student uses the above fact to obtain

$$\frac{1}{x+\sqrt{x^2+1}} + \frac{1}{y+\sqrt{y^2+1}} = 2020(x+y)$$

or another useful intermediate result.

1 point: Student rewrites the left side using a common denominator.

1 point: Student uses the given information that $\sqrt{x^2+1}+\sqrt{y^2+1} = 2021(x+y)$ to obtain

$$\frac{2022(x+y)}{\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)} = 2020(x+y)$$

or another useful intermediate result.

1 point: Student manipulates the preceding equation to arrive at the correct answer of $\frac{1011}{1010}$. Don't withhold this point if the student forgets to reduce the fraction.

Note: Award **1 point** if the student writes the correct answer with no explanation, or if the student obtains the correct answer for a specific situation (e.g., $x = 0$) without additional constructive progress towards proving the answer in general.

Note: Some students divided by $x + y$ or another algebraic expression without first proving that the expression is nonzero. We didn't deduct points for this, but it is important to be mindful of this when manipulating algebraic expressions.

Note: Some students used Wolfram Alpha to trivialize the problem. If the student shows their Wolfram Alpha inputs and outputs, the contest rules require that we give the student **5 points**. If the student doesn't show the Wolfram Alpha inputs and outputs, don't give any points for the results obtained using Wolfram Alpha, but the student should still receive **1 point** for the correct answer.

Problem 4/2/33:

1 point: Student includes a good diagram that builds on the given information. Don't automatically withhold this point if the student doesn't include a diagram, but if the absence of a good diagram makes the solution hard to follow, withhold this point.

1 point: Student recognizes that if $\alpha = 0$, we may select $P \equiv O$, and/or if $\alpha = 1$, we may select $P \equiv I$.

3 points: Student shows that if $\alpha \neq 0, 1$, P lies on line IO. Award partial credit as appropriate. Any meaningful observation that is reasonably well connected to what the student is trying to show should receive at least **1 point** of partial credit.

Note: Award **4 points** if the student proves the converse of the problem statement, but doesn't explain why this means that the original problem statement is true.

Problem 5/2/33:

Note: Showing that the correct answers work is worth **2 points** and showing that no other answers are possible is worth **3 points**.

Note: Students can cite the "Frobenius Coin problem," "Chicken McNugget theorem," or any similar result without proof. Students still need to explain how they are applying these results to the present problem.

Note: Award **1 point** if the student writes the correct answer with no explanation.

2 points: Student shows that $cc(G) = cc(\{a, b\}) \cup \{b\}$, and applies this result to show that $(a, b) = (2, 2k + 1)$, $(a, b) = (3, 3k + 1)$, or $(a, b) = (3, 3k + 2)$, where k is any positive integer, satisfies the conditions in the problem. Award **1 point** of partial credit for significant constructive progress towards this result, such as using $G = \{a, a+b, 2b\}$ and obtaining a meaningful intermediate result.

3 points: Student shows that $a \geq 4$ gives no additional answers. Award partial credit as appropriate for significant constructive progress towards this result. To get **1 point** of partial credit, it is sufficient to use the given properties to show that the m in property (3) must be one of a or b. The intermediate result that $a + b$ must be in G is also worth **1 point** of partial credit. If the student obtains different but similarly useful intermediate results, award partial credit based on how close the student is to completing a valid solution.

Note: Award **1 point** if the student finds one of the two classes of solutions (e.g., $G = (2, b+2)$, but not $G = (3, b+3, 2b)$). Award additional credit as appropriate for constructive progress towards proving that $a > 3$ is impossible.

Note: Award **1 point** if the student correctly characterizes $cc(\{a, b\})$ (e.g., "If a and b are relatively prime, every integer can be uniquely written in the form $ma+nb$ for $0 \leq n < a$. The impossible sums are those in which m is negative."). Using the Frobenius Coin theorem to say that the largest impossible sum is $ab - a - b$ is not sufficient to receive this point.