USA Mathematical Talent Search
Round 1 Grading Rubric
Year 28 — Academic Year 2016–2017
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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

IMPORTANT NOTE: Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

Problem 1/2/28:

Award **5 points** for the correct configuration of shaded regions. No justification is required. Withhold **1 point** for each *region* that is shaded (or unshaded) incorrectly.

Note: Award **4 points** if the configuration is correct, except the shaded and unshaded regions are reversed.

Note: Withhold **1 point** if the presentation is poor (e.g., regions are shaded in yellow so the shading isn't visible when the solutions are printed in black and white, the diagram is very messy, the diagram is split across multiple pages).

Problem 2/2/28:

Note: Solutions by exhaustive search (e.g., a computer program that analyzes all possible values of x, y, and z) are disallowed for this problem as this would make the problem too easy; a solution by exhaustive search should receive **0 points**. For this reason, students who write the correct answer, but do not provide any explanation of how they found the answer should typically receive **0 points**. If, however, the student demonstrates that the correct triple satisfies all the conditions in the problem but provides no other meaningful work, award **1 point**.

1 point: Student applies the condition that x, y, and z form an arithmetic progression AND the condition that x, y, and $z + 1000$ form a geometric progression to come up with useful expressions for x, y, and/or z.

1 point: Student obtains the equation $(z - x)^2 = 4000x$.

1 point: Student recognizes that $x = 10n^2$ for some positive integer n.

1 point: Student explains how the requirement that x, y, and z are three-digit positive integers constrains the possible values of n. To receive this point, it is sufficient to explain either why the fact that x is a three-digit number means that $n \geq 4$ or why the fact that z

is a three-digit number means that $n \leq 4$.

1 point: Student completes the solution and proves that $(x, y, z) = (160, 560, 960)$ is the only triple that works.

Problem 3/2/28:

Note: Many students did not realize that we intended for students to give a general answer in terms of m and n. Thus, many students instead found that the numerical value of d is maximized over all m and n for an equilateral triangle and a square, which gives an answer of $d = 15^\circ$. If a student gives an answer of $d = 15^\circ$, they should receive full credit **(5 points)** if they (1) prove the general formula of $d = \frac{180^\circ}{mn}$ and then note that we get the maximum numerical value of d when $m = 3$ and $n = 4$ (or vice versa), or (2) prove that the numerical value of d is maximized when $m = 3$ and $n = 4$.

A student who shows that $d = 15^\circ$ when $m = 3$ and $n = 4$ without providing a proof that this gives the maximum numerical value of d over all m and n should receive **1 point**. Additional credit should be awarded if the student makes constructive progress towards a proof that the case when $m = 3$ and $n = 4$ gives the maximum numerical value of d. Award an additional **1 point** if the student does a partial analysis of additional numerical values of m and n that need to be considered.

For students using a method similar to the official solution, award points as follows (see official solution for a description of the notation):

1 point: Student proposes that if we consider each point on the circle as a number from 0 to 1, then we can let each term of the m-progression be an integer multiple of $\frac{1}{mn}$ and let each term of the n-progression be a half-integer times $\frac{1}{mn}$.

1 point: Student recognizes (with explanation) that to complete the proof, it will be worthwhile to show that $\left|d - \frac{1}{mn}\right|$ is a difference between terms of the progression.

1 point: Student applies Bezout's lemma to show that there are integers p, q such that $pm - qn = 1$.

1 point: Student recognizes that $\left(x + \frac{q}{m}\right) - \left(y + \frac{p}{n}\right) = d - \frac{1}{mn}$.

1 point: Student completes the proof by recognizing that based on the definition of d, we know that $d \leq \left|d - \frac{1}{mn}\right|$, which implies that $d \leq \frac{1}{2mn}$. This point should only be awarded if the student receives the preceding two points involving applying Bezout's lemma and showing that $\left(x + \frac{q}{m}\right) - \left(y + \frac{p}{n}\right) = d - \frac{1}{mn}$.

Problem 4/2/28:

0 points: Student describes an incorrect coding scheme.

1 point: Student describes a correct coding scheme, but does not explain why it works.

2 points: Student describes a correct but highly inefficient coding scheme and explains why it works.

3 points: Student describes a correct and reasonably efficient (but not optimal) coding scheme and explains why it works. To receive **3 points**, the student must have $n < 5000$ and there must not be an easy way (within the student's method) to make the coding scheme significantly more efficient.

5 points: Student describes an optimal coding scheme ($n = 2026$) and explains why it works.

Problem 5/2/28:

1 point: Student recognizes that the second condition implies that there exists a quadratic $f(x) = ax^2 + bx + c$ such that $y_k = f(k)$ for each k.

1 point: Student applies the first condition to get the following system of three equations:

$$\begin{aligned} ap_2 + bp_1 + cp_0 &= 0, \\ ap_3 + bp_2 + cp_1 &= 0, \\ ap_4 + bp_3 + cp_2 &= 0. \end{aligned}$$

1 point: Student calculates $\det(M)$ in terms of the p_i.

1 point: Student applies the formulas for p_0, p_1, p_2, p_3, and p_4 in terms of $p_0 = n$ to calculate $\det(M)$ in terms of n.

1 point: Student completes proof by explaining why M is invertible, enabling us to conclude that $a = b = c = 0$, and therefore all the y_k are 0.

Note: Analyzing the $n = 4$ case by itself is not sufficient to receive any points.