USA Mathematical Talent Search

Round 1 Grading Rubric

Year 28 — Academic Year 2016–2017

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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

IMPORTANT NOTE: Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

Problem 1/1/28:

Award **5 points** for the correct configuration of numbers. No justification is required. Withhold **1 point** for each incorrect entry.

Problem 2/1/28:

Note: Since the problem states "*if* when the squares of the tower are colored black and white ..." (emphasis added), the uniqueness of a tower depends only on the spatial configuration of the squares, and not on the coloring scheme. That is, whether the leftmost square in the bottom row is colored black vs. white does not affect the number of balanced towers. Some students multiplied the correct answer of $2^{1007}\binom{1008}{504}$ by 2 since the leftmost square in the bottom row could be colored either black or white. This *is* a misinterpretation of the problem; however, it is a relatively minor (and subtle) error. Thus, students who submitted otherwise excellent solutions were eligible to receive **5 points** on this problem. However, if we thought that a solution was otherwise "in between scores" and a student made this error, we rounded down. Points were awarded as follows:

1 point: Student recognizes that each even row has the same number of black and white squares and each odd row has a number of black and white squares differing by 1.

1 point: Student recognizes that in order for a tower to be balanced, the number of rows with one more black square must equal the number of rows with one more white square.

3 points: Student uses constructive counting or another valid method to get the correct answer of $2^{1007}\binom{1008}{504}$. Partial credit **(1 or 2 points)** should be awarded for constructive progress towards the answer. To receive **2 points** of partial credit, students must describe both a method for counting the number of options for the odd rows and a method for counting the number of options for the even rows, and at least one of these counting methods must be correct. The correct answer itself is worth **1 point**; no justification is required to receive this point.

Problem 3/1/28:

1 point: Student plugs in values such as $x = 2$, $y = -1$, and $z = -1$ to get a useful expression.

2 points: Student provides a complete analysis of why $n = 1$ is the only odd value of n that works. Demonstrating that $n = 1$ works is sufficient to receive **1 point** of partial credit.

2 points: Student provides a complete analysis of why $n = 4$ is the only even value of n that works. Demonstrating that $n = 4$ works is sufficient to receive **1 point** of partial credit.

Note: A student who provides the correct answer without any explanation should not receive any credit for this problem.

Problem 4/1/28:

1 point: Student recognizes (with or without justification) that one solution is $f(x) = x^2$. Award this point even if the student also proposes that other functions are solutions.

4 points: Student provides a mathematically rigorous justification of why no other functions (including non-polynomial functions) satisfy all the constraints in the problem. Student solutions varied considerably, so the amount of partial credit depended in large part on how far or close a student solution was to a complete solution. We tended to be conservative with respect to awarding partial credit, especially for students who did not consider non-polynomial functions. For students using a method similar to the official solution, these four points were awarded as follows:

1 point: Student applies the constraint that $f(x) \le Cx^2$ to propose something useful, such as seeing what happens if for some t, we have $\frac{f(t)}{t^2} = a \ne 1$.

1 point: Student recognizes that we get a sequence $\langle t_n \rangle$ with $t_0 = t$ such that

$$f(t_n) = a^{(-2)^n} t_n^2,$$

or the student makes other equivalent constructive progress.

2 points: Student analyzes the sequence to conclude that no other functions satisfy all the constraints in the problem. Award **1 point** (out of these 2 points) for significant constructive progress towards completing the proof (e.g., recognizing that $a^{(-2)^n}$ is unbounded for any $a \ne 0$ or 1).

Problem 5/1/28:

Note: Student solutions varied considerably, so we used various benchmarks to determine how many points to award (assuming a clear and well written solution).

5 points: Student provides a complete and correct proof that the maximum possible area of $ABCD$ is $\frac{9}{32}$.

4 points: Student shows that it is possible for $ABCD$ to have an area of $\frac{9}{32}$, and the student makes significant constructive progress towards proving that this is the maximum possible area.

3 points: Student shows that it is possible for $ABCD$ to have to have an area of $\frac{9}{32}$, but the student does not prove that this is the maximum possible area.

2 points: Student obtains an upper bound (larger than $\frac{9}{32}$) for the maximum possible area of $ABCD$.

1 point: Student uses the formula $Area = \frac{1}{2}ab\sin C$ or applies another relevant theorem or trig formula, but doesn't make considerably more constructive progress. Award additional points if the student makes additional constructive progress.

1 point: Student writes the answer $\frac{9}{32}$. At least some justification must be provided to earn this point.

For the students who used a method similar to the official solution, the five points were awarded as follows:

1 point: Student applies Varignon's theorem to find that $PQRS$ is a parallelogram,

1 point: Student recognizes that $PR \le \frac{AD+BC}{2}$ or $QS \le \frac{AB+CD}{2}$ (or both).

1 point: Student recognizes that $x + y \le \frac{5}{8}$ or $x^2 + y^2 = \frac{1}{4}$ (or both).

1 point: Student recognizes that $2xy \le \frac{9}{64}$.

1 point: Student completes proof and recognizes that the maximum possible area of $ABCD$ is $\frac{9}{32}$.

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