USA Mathematical Talent Search

Round 2 Grading Criteria

Year 25 — Academic Year 2013–2014

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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct 1 additional point for a solution that is poorly written.

Problem 1/2/25:

5 points for a correct configuration. Withhold a point for each grid cell that was filled with an incorrect number.

If a configuration is not fully correct, you should specifically point out at least one error in your comments. A correct proof of uniqueness should be commended in the comments.

Problem 2/2/25:

There are 2 typical approaches to this problem. First is as the given solution: dissect the figure into triangles and note that $\triangle EOC$ is similar to $\triangle FCO$ (points F and E are not defined in the prompt). The second is to use the circle to determine the height of the trapezoid and the sum of the side lengths and the set up a couple of Pythagorean Theorem equations.

Notes:

- Students may use that $AB + CD = BC + DA$ without proof.
- Since a diameter meets AB and CD, the altitude from B to CD intersects segment CD instead of intersecting the line outside the segment. Likewise for the altitude from A. Some students may have assumed this fact in their proofs. Deduct up to 2 points from a student who uses this without proof in an integral way.

Give 1/5 for a solution that makes any nontrivial progress. Give at least 2/5 for any solution that drops tangents from the center to the sides. Give 4/5 for any solution that is mostly complete with a minor error. Give 5/5 for a complete and correct solution.

Problem 3/2/25:

Almost every student solved this problem by cases on the parity of n. There are 3 standard approaches, all using finite geometric series to rationalize the denominator.

The given solution uses $\left(-\sqrt[n]{2}\right)^{2n} = 4$, so

$$\frac{1-4}{1+\sqrt[n]{2}} = \frac{1-\left(-\sqrt[n]{2}\right)^{2n}}{1-(-\sqrt[n]{2})} = -\sum_{i=0}^{2n-1}(-\sqrt[n]{2})^i$$

From here we get

$$P(x) = \boxed{\frac{1}{3}\sum_{i=0}^{2n-1}(-x)^i}.$$

Alternatively, we could use $\left(-\sqrt[n]{2}\right)^n = \pm 4$ and need casework on the parity of n. The approach is the same, but the polynomials become

$$P_{\text{even}}(x) = \boxed{-\sum_{i=0}^{n-1}(-x)^i}$$
$$P_{\text{odd}}(x) = \boxed{\frac{1}{3}\sum_{i=0}^{n-1}(-x)^i}.$$

The coefficients are $\frac{1}{1\pm 2}$.

The third solution decomposes $n = 2^a b$ with b odd. For this solution we need to use difference of squares a times:

$$\begin{aligned}1 - y^{2^a b} &= (1 + y^{2^{a-1}b})(1 - y^{2^{a-1}b})\\ &= (1 + y^{2^{a-1}b})(1 + y^{2^{a-2}b})(1 + y^{2^{a-2}b})\\ &\vdots\\ &= (1 + y^{2^{a-1}b})(1 + y^{2^{a-2}b})\cdots(1 + y^b)(1 - y^b),\end{aligned}$$

and then $1 - y$ is a factor of the final term. If $y = -\sqrt[n]{2}$ then we get 1 ± 2 equal to some expression with $1 - (-\sqrt[n]{2})$ as a factor. If we set $y = -x$ and divide by $1 + x$ we get

$$P_{\text{large}}(x) = \boxed{\frac{1}{1\pm 2}(1-(-x)^{2^{a-1}b})(1-(-x)^{2^{a-2}b})\cdots(1-(-x)^b)\cdot(1+(-x)+\cdots+(-x)^{n-1})}.$$

The sign on 1 ± 2 is negative if $a > 0$ and positive if $a = 0$.

Give at least 1 point to any student who shows any insight that finite geometric series can be used to solve this problem.

Problem 4/2/25:

- 1 point for a nontrivial observation or manipulation of the given conditions.
- 2-3 points on partial progress for a solution.
- 4-5 points for an essentially correct solution. In particular be aggressive for giving 4 points for a "correct" solution that is difficult to follow.

Most solutions will show that the given expression is less than 12072, and this is a strong signal that the solution is correct. Solutions that show that the given expression is less than some number less than 12072 should be viewed with some skepticism.

It seems that the second given inequality (the one that starts with n^2a_1) is unnecessary, so the fact that a solution doesn't use it is no cause for concern.

Problem 5/2/25:

Part (a) is worth 3 points and part (b) is worth 2 points.

Many solutions break down into two main parts:

- Constructing configurations of tiles that may be used as building blocks to form more complex configurations. S_n in the official solution is one such example. Students will come up with many different constructions for their building blocks. Deduct 1-2 points if students do not make it explicitly clear how their configurations are being constructed. For instance, a student might define S_n in part (a) by drawing S_2, S_3, and S_4, and leaving it to the reader to determine the general method of construction.
- Developing operations that connect these building blocks so that the number of domino-tilings of the resulting configuration is a sum or product of the number of domino-tilings of the building blocks. Award 2 points to any student that managed to do this, even if he or she did not get correct solutions to parts (a) and (b).

Students may use well-known results about Fibonacci numbers such as Zeckendorf's Theorem without citation. They also do not need to prove that the number of domino-tilings of a $2 \times n$ region is the nth Fibonacci number.