USA Mathematical Talent Search

Round 3 Problems

Year 36 — Academic Year 2024-2025

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **January 6, 2025** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on January 6, 2025.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 503
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before January 6, 2025.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "Account" page.

6. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My Scores". You will also receive an email when your scores and comments are available (provided that you did item #5 above).

These are only part of the complete rules.
Please read the entire rules at www.usamts.org.

USA Mathematical Talent Search

Round 3 Problems

Year 36 — Academic Year 2024-2025

www.usamts.org

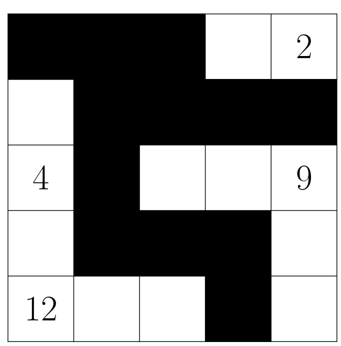
Each problem is worth 5 points.

1/3/36. Shade some squares in the grid so that:

1. Squares with numbers are unshaded.
2. Each number is equal to the product of the number of unshaded squares it can "see" in its row and column. (A square can see another square if they're in the same row or column and the sight line between them doesn't have any shaded squares. Each square can see itself.)
3. The shaded squares must make one connected group. Two squares are considered to be connected if they share an edge.

									4
			6			6			
		36	24						
				16					
					24				
						36	18		
			6			12			
36									

The following is an example of a completed puzzle to clarify the rules.



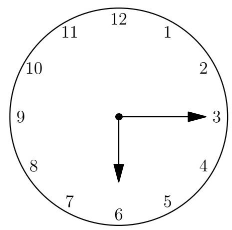
There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

(The problems are continued on the next page.)

USA Mathematical Talent Search
Round 3 Problems
Year 36 — Academic Year 2024-2025
www.usamts.org

2/3/36. Calamitous Clod deceives the math beasts by changing a clock at Beast Academy. First, he removes both the minute and hour hands, then places each of them back in a random position, chosen uniformly along the circle.

Professor Grok notices that the clock is not displaying a valid time. That is, the hour and minute hands are pointing in an orientation that a real clock would never display. One such example is the hour hand pointed at 6 and the minute hand pointed at 3.



The math beasts can fix this, though. They can turn both hands by the same number of degrees clockwise. On average, what is the minimal number of degrees they must turn the hands so that they display a valid time?*

**Assume that after Calamitous Clod replaces the hands, they don't move again until the math beasts adjust their position.*

3/3/36. Let a, b be positive integers such that $a^2 \geq b$. Let

$$x = \sqrt{a + \sqrt{b}} - \sqrt{a - \sqrt{b}}.$$

(a) Prove that for all integers $a \geq 2$, there exists a positive integer b such that x is also a positive integer.

(b) Prove that for all sufficiently large a, there are at least two b such that x is a positive integer.

Note: We've received some questions about what is meant by "for all sufficiently large a." To give a simple example of this phrasing, it is true that for all sufficiently large positive integers n, we have $n^2 \geq 100$. Specifically, this is true for all $n \geq 10$.

4/3/36. $ABCD$ is a convex quadrilateral where $\angle A = 45^\circ$ and $\angle C = 135^\circ$. P is a point strictly inside $\triangle ABC$ such that $\angle BAP = \angle CAD$ and $\angle BCP = \angle ACD$. Prove that $\overline{PB} \perp \overline{PD}$ if and only if $\overline{AC} \perp \overline{BD}$.

5/3/36. Find all ordered triples of nonnegative integers (a, b, c) satisfying $2^a \cdot 5^b - 3^c = 1$.

USA Mathematical Talent Search
Round 3 Problems
Year 36 — Academic Year 2024-2025
`www.usamts.org`

Problems by Josiah Balete, Emil Rumney, and USAMTS Staff.
Round 3 Solutions must be submitted by **January 6, 2025**.
Please visit **https://www.usamts.org** for details about solution submission.